This activity is about finding connections between exponential functions and their rates of change. This will involve drawing tangents to graphs and using spreadsheets.

Information sheet A Exponential functions and graphs

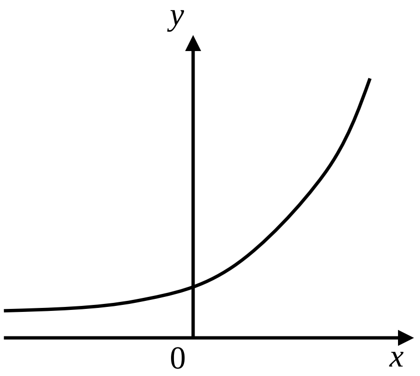
There are many examples of exponential growth and decay in everyday life, such as bacteria growth and radioactive decay. Population growth depends on the size of the current population. The rate of population growth can be used to predict the size of the population at any given time.

The number of people in the population, N at any time t, can be written as an exponential function of the form:

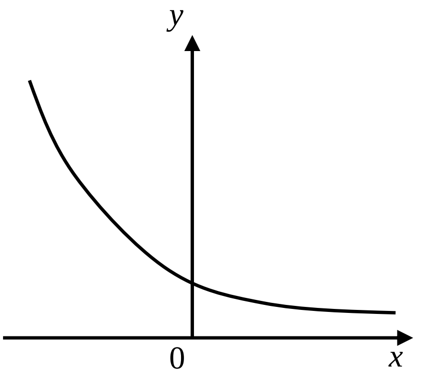
$N = Ae^{rt}$ where A and r are constants

Graphs of exponential growth and decay have the distinctive shapes shown below.

Exponential growth curve



Exponential decay curve



Think about …

Can you suggest any everyday examples of exponential growth or decay?

What can you say about the gradient of the graph showing exponential growth?

What can you say about the gradient of the graph showing exponential decay?

Try these A

1a Use your calculator to complete the $y = e^x$ row in the table below, giving values to 1 dp.

x	-2	-1	0	1	2
$y = e^x$					
Gradient					

Check that your values agree approximately with points lying on the curve shown here.

b Draw tangents to the curve at the points given in the table.

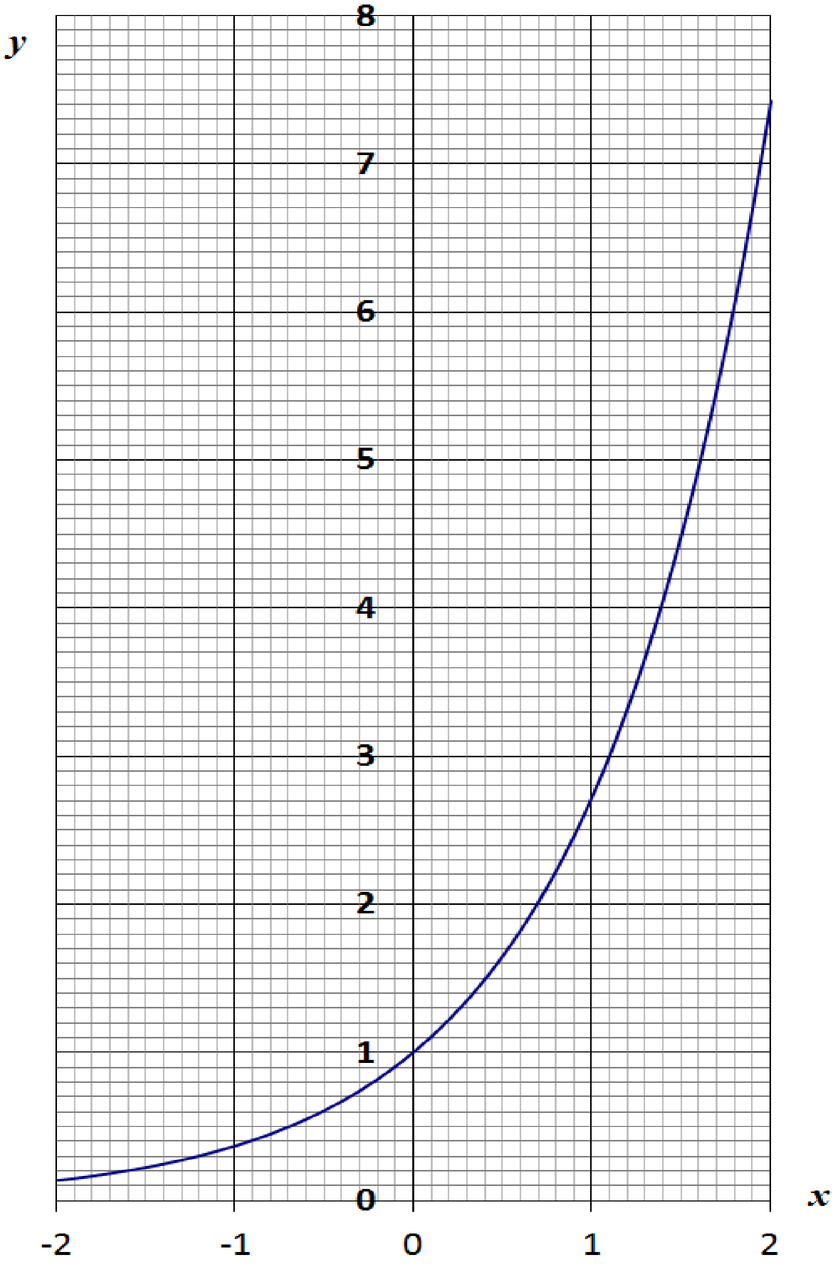
Find the gradient of each tangent and write the value, correct to 1 dp, in the table.

Think about...

Compare the values in the last two rows of the table.

Do you notice anything?

Graph of $y = e^x$



2a Use your calculator to complete the $y = e^{2x}$ row in the table below, giving values to 1 dp.

x	–2	–1	0	1	2
$y = e^{2x}$					
Gradient					

Check that your values give points lying on the curve shown below.

b Draw tangents to the curve at the points given in the table.

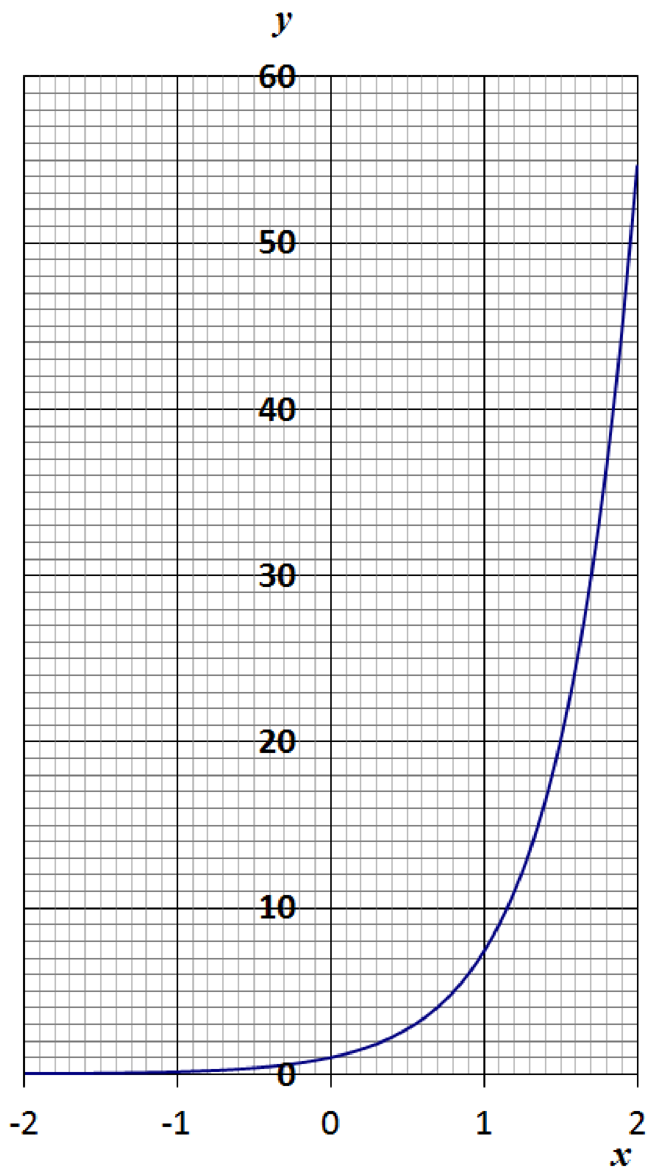
Find the gradient of each tangent and write the value, correct to 1 dp, in the table.

Think about...

Compare the values in the last two rows of the table.

Do you notice anything?

Graph of $y = e^{2x}$



3a Use your calculator to complete the $y = e^{-x}$ row in the table below, giving values to 1 dp.

x	-2	-1	0	1	2
$y = e^{-x}$					
Gradient					

Check that your values agree approximately with points lying on the curve shown below.

b Draw tangents to the curve at the points given in the table.

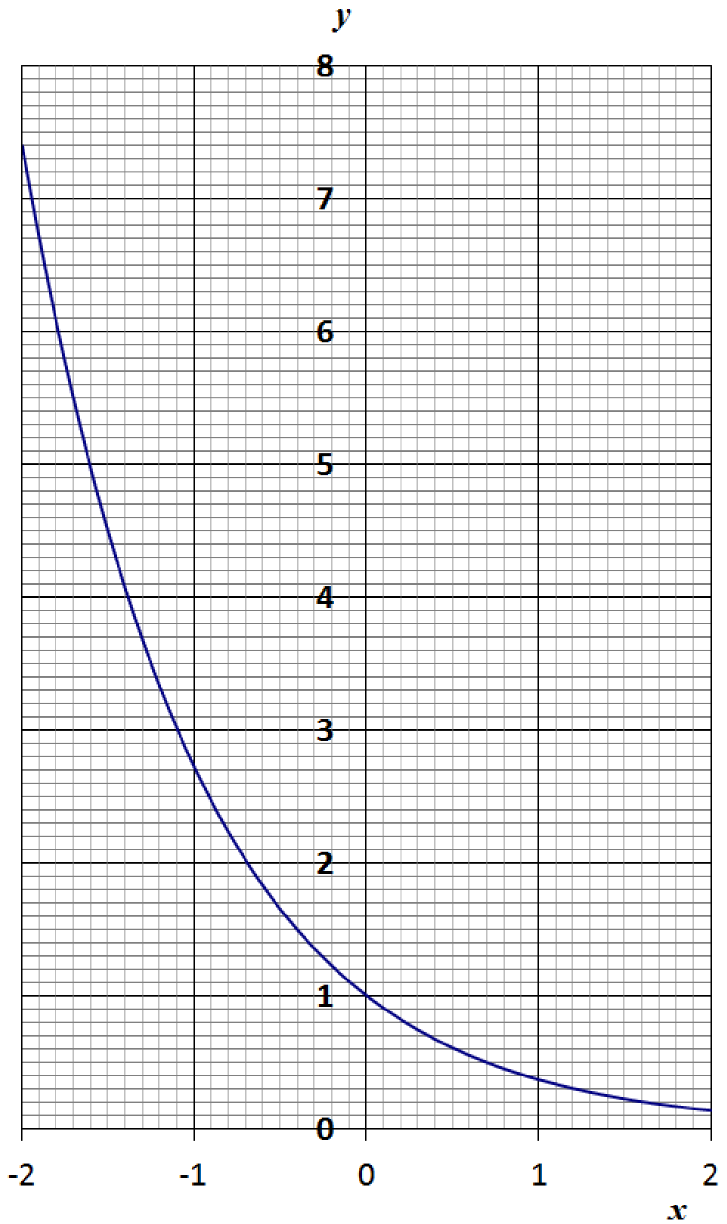
Find the gradient of each tangent and write the value, correct to 1 dp, in the table.

Think about...

Compare the values in the last two rows of the table.

Do you notice anything?

Graph of $y = e^{-x}$

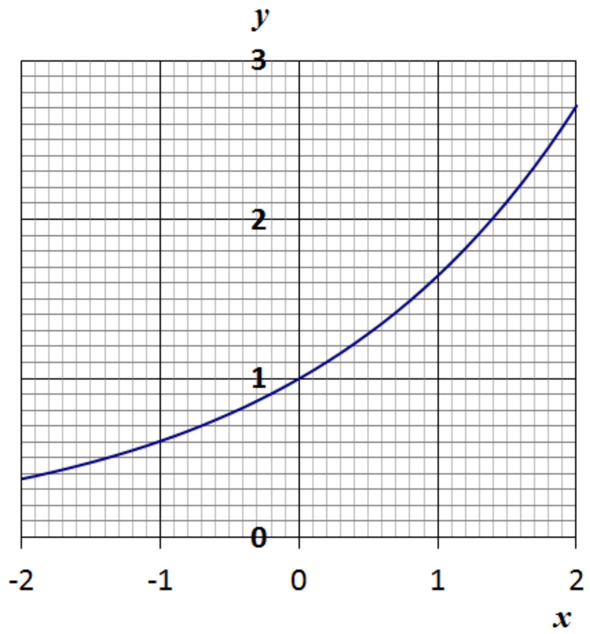


4a Use your calculator to complete the $y = e^{0.5x}$ row in the table below, giving values to 1 dp.

x	–2	–1	0	1	2
$y = e^{0.5x}$					
Gradient					

Check that your values agree approximately with points lying on the curve shown below.

Graph of $y = e^{0.5x}$



b Draw tangents to the curve at the points given in the table.

Find the gradient of each tangent. Write the value, correct to 1 dp, in the table.

Think about...

Compare the values in the last two rows of the table.

Do you notice anything?

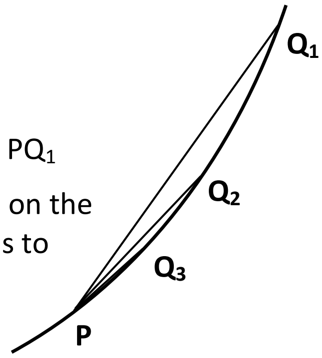
Think about...

Use your findings to predict a connection between the y values for the graph of $y = e^{mx}$ and the values of the gradient function.

Information sheet B Working out gradients

Finding the gradient of curves by drawing tangents by hand is not a very accurate method.

Better results can be achieved by calculation.



The sketch shows a point P on a curve.
Suppose that Q_1 is a second point on the curve near to P.
The co-ordinates of P and Q_1 can be used to find the gradient of the chord PQ_1

Other points Q_2 and Q_3 lying on the curve even nearer to P are also shown on the sketch. The nearer that the point Q is to P, the nearer the gradient of PQ is to the gradient of the tangent at P.

In general, the gradient at a point P where $x = a$,
on the curve $y = \mathrm{f}(x)$ is estimated using:

gradient $\approx \dfrac{\mathrm{f}(a+h)-\mathrm{f}(a)}{h}$ where h is a small increment.

In terms of the sketch,
h represents the difference in the x coordinates at P and Q
and $f(a + h) - f(a)$ represents the difference in the y coordinates of P and Q.

A spreadsheet can be used to estimate the gradient at a number of points on a curve.

Try these B

1 The gradient function of $y = e^x$

The spreadsheet below gives formulae that can be used to estimate gradients on the curve $y = e^x$.

The formulae in column A work out x coordinates at intervals of 0.1, starting with $x = -2$

The formulae in column B work out the corresponding y coordinates.

The formulae in column C estimate the gradient of the curve at each point, using an increment of 0.01.

	A	B	C
1	x	$\mathbf{f}(x) = \mathbf{e}^x$	**Gradient**
2	-2	=EXP(A2)	=(EXP(A2+0.01)-EXP(A2))/0.01
3	=A2+0.1	=EXP(A3)	=(EXP(A3+0.01)-EXP(A3))/0.01
4	=A3+0.1	=EXP(A4)	=(EXP(A4+0.01)-EXP(A4))/0.01
5	=A4+0.1	=EXP(A5)	=(EXP(A5+0.01)-EXP(A5))/0.01

Copy these formulae onto a spreadsheet, using 'fill down' to extend the results to $x = 2$.

Think about…

Compare the values found in columns B and C. What do you notice?

2 The gradient function of $y = e^{2x}$

The spreadsheet below shows formulae that can be used to estimate gradients on the curve $y = e^{2x}$.

	A	B	C
1	x	$f(x) = e^{2x}$	Gradient
2	-2	=EXP(2*A2)	=(EXP(2*(A2+0.01))-EXP(2*A2))/0.01
3	=A2+0.1	=EXP(2*A3)	=(EXP(2*(A3+0.01))-EXP(2*A3))/0.01
4	=A3+0.1	=EXP(2*A4)	=(EXP(2*(A4+0.01))-EXP(2*A4))/0.01
5	=A4+0.1	=EXP(2*A5)	=(EXP(2*(A5+0.01))-EXP(2*A5))/0.01

Copy these formulae onto another worksheet, and use 'fill down' to extend the results to $x = 2$.

Compare the values found in columns B and C. What do you notice this time?

Use the spreadsheet to draw graphs of $y = e^{2x}$ and its gradient function on the same axes.

Compare the curves and write down what you notice.

3 The gradient function of $y = e^{0.5x}$

Make a copy of the worksheet you used for $y = e^{2x}$.

Find values for $y = e^{0.5x}$ and its gradient function by replacing '2' in cells B1, B2 and C2 by '0.5' (leaving A2 unchanged).

Use 'fill down' to change the other cells in columns B and C and extend the table to $x = 2$.

Again compare the values in columns B and C and draw graphs of $y = e^{0.5x}$ and its gradient function on the same axes. Write down what you notice.

4 The gradient function of other functions of the form $y = e^{kx}$

Repeat the process for $y = e^{-x}$ and other exponential functions of the form $y = e^{kx}$ where k is any constant.

Extension

Investigate the gradient functions of functions of the form $y = ae^{kx}$, $y = e^{kx + c}$ and $y = ae^{kx + c}$ where a, k and c are constants.

Reflect on your work

- Compare the two methods you have used for finding the gradients of exponential functions. What are their advantages and disadvantages?
- What can you say in general about the gradient function of $y = e^{kx}$?